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CONTACT PROBLEM FOR A NARROW ANNULAR PUNCH. UNKNOWN REGION OF CONTACT

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The nonlinear problem of determining the contact stresses and the contact zone under the base of a narrow annular punch is studied. An asymptotic model of one-sided contact along the line is constructed by the method of matched asymptotic expansions. Explicit asymptotic formulas for the line-pressure density are obtained. The asymptotic representation of the contact arc is given.

1. Formulation of the Problem and Discussions. Let Γ be the circumference of radius R determined by the polar angle θ (see Fig. 1). We consider a ring $\Gamma(\varepsilon)$ whose middle line is Γ and denote its half-width and relative half-width by h and $\varepsilon = h/R$, respectively. The displacement vector $\boldsymbol{u} = (u_1, u_2, u_3)$ of the elastic half-space under the action of an annular punch with the base $\Gamma(\varepsilon)$ satisfies the problem

$$\mu\Delta_x \boldsymbol{u}(\varepsilon;\boldsymbol{x}) + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u}(\varepsilon;\boldsymbol{x}) = 0, \quad x_3 < 0; \tag{1.1}$$

$$\sigma_{31}(\boldsymbol{u};\boldsymbol{x}) = \sigma_{32}(\boldsymbol{u};\boldsymbol{x}) = 0, \quad x_3 = 0; \tag{1.2}$$

$$\sigma_{33}(\boldsymbol{u};\boldsymbol{x}) = 0, \qquad x_3 = 0, \qquad (x_1,x_2) \notin \Gamma(\varepsilon); \tag{1.3}$$

$$\begin{gathered} u_3(\varepsilon;\boldsymbol{x}) \leqslant -\delta_0 - \beta_2 x_1, \qquad \sigma_{33}(\boldsymbol{u};\boldsymbol{x}) \leqslant 0, \\ [u_3(\varepsilon;\boldsymbol{x}) + \delta_0 + \beta_2 x_1]\sigma_{33}(\boldsymbol{u};\boldsymbol{x}) = 0, \quad x_3 = 0, \quad (x_1,x_2) \in \Gamma(\varepsilon); \end{gathered} \tag{1.4}$$

$$\boldsymbol{u}(\varepsilon;\boldsymbol{x}) = o(1), \qquad |\boldsymbol{x}| = (x_1^2 + x_2^2 + x_3^2)^{1/2} \to \infty. \tag{1.5}$$

Here Δ_x is the Laplace operator, λ and μ are the Lamé parameters, $\sigma_{3j}(\boldsymbol{u})$ are the stress-tensor components, and δ_0 and β_2 are the translation of the punch and its rotation about the Ox_2 axis, respectively.
Expressing the normal component of the displacement vector at the boundary point in terms of the surface pressure p and denoting Young's modulus and the Poisson ratio by E and ν, respectively, we write the conditions of one-sided contact (1.4) in the form

$$\begin{gathered} p(x_1,x_2) > 0 \Rightarrow \iint\limits_{\Gamma(\varepsilon)} \frac{p(\xi_1,\xi_2)}{\sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2}}\, d\xi_1\, d\xi_2 = \frac{\pi E}{1-\nu^2}\,(\delta_0 + \beta_2 x_1), \\ p(x_1,x_2) = 0 \Rightarrow \iint\limits_{\Gamma(\varepsilon)} \frac{p(\xi_1,\xi_2)}{\sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2}}\, d\xi_1\, d\xi_2 \geqslant \frac{\pi E}{1-\nu^2}\,(\delta_0 + \beta_2 x_1), \\ p(x_1,x_2) \geqslant 0, \qquad (x_1,x_2) \in \Gamma(\varepsilon). \end{gathered} \tag{1.6}$$

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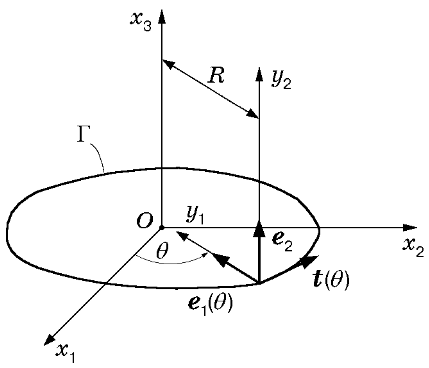

Fig. 1

Remark 1. It follows from relations (1.6) that the contact cannot occur at the points $(x_1, x_2) \in \Gamma(\varepsilon)$, at which the punch surface is above the undisturbed boundary of the elastic base.

Formulated within the framework of the linear theory of elasticity, problem (1.1)–(1.5) is nonlinear. Since the calculation scheme of the structure changes upon loading (the new constraints arise or the existing constraints fail), the nonlinearity of this kind is called the structural nonlinearity. The mathematical aspects of these problems were studied by methods of the theory of variational inequalities (see, e.g., [1, 2]). Gol'dshtein and Spektor [3] developed methods of qualitative analysis. Numerical algorithms were proposed in [1, 2, 4]. Khludnev [5] studied the problems of optimal control. Nazarov and Polyakova [6] derived an inequality similar to that considered in the present study and used it to formulate the fracture criteria in the mechanics of cracking.

The distinguishing feature of the above-formulated problem is the need to indicate when the type of boundary condition in (1.4) should be changed. The unknown zone of contact is a narrow subregion of the ring $\Gamma(\varepsilon)$, which contracts to a certain arc of circumference Γ as ε decreases. In this case, we will deal with the contact along the line [7]. The stage-by-stage process of refining the boundaries of the region on $\Gamma(\varepsilon)$ where the contact between the punch and the elastic base occurs is called the contact-zone variation [8].

Below, we consider the asymptotic representations used to construct an approximate solution of problem (1.1)–(1.5). They were first employed in [7, 9, 10]. The method of matched expansions used in Sec. 3 is outlined in detail in [11]. The aim of the present study is to obtain compact formulas suitable for calculations. Argatov and Nazarov [8] treated a similar problem for the Laplace operator and substantiated the asymptotic solution. The assumption that friction is absent allows one to use the Papkovich–Neuber representation and the results of [8] to solve the problem posed.

2. External Asymptotic Representation. We denote the solution of the Boussinesq problem of an elastic half-space $x_3 \leqslant 0$ subjected to a unit point force applied at the origin of coordinates and directed oppositely to the Ox_3 axis by $\boldsymbol{T}$. The displacement vector of the half-space subjected to the load distributed along the contour Γ with a certain line density P is given by

$$\boldsymbol{v}(P; \boldsymbol{x}) = \int_{-\pi}^{\pi} P(\tau)\boldsymbol{T}(x_1 - R\cos\tau,\ x_2 - R\sin\tau,\ x_3)R\,d\tau. \tag{2.1}$$

The vector function (2.1) serves as an approximation to the solution $\boldsymbol{u}$ in the region remote from $\Gamma(\varepsilon)$ if P is interpreted as the intensity of contact pressures per unit length of the arc of the middle line of the punch base.

In the neighborhood of Γ, we introduce the curvilinear coordinates $(\theta,\ y_1,\ y_2)$ (see Fig. 1) related to the Cartesian coordinates by the formulas

$$x_1 = (R - y_1)\cos\theta, \qquad x_2 = (R - y_1)\sin\theta, \qquad x_3 = y_2. \tag{2.2}$$

The Cartesian components of the vector of the displacements caused by the unit force applied at the point on Γ with the angular coordinate τ are written in the form

$$T_1(\theta, \boldsymbol{y}; \tau) = -\frac{R(\cos\theta - \cos\tau) - y_1 \cos\theta}{4\pi\mu R_{\boldsymbol{y}}(\theta,\tau)} \Big(\frac{y_2}{R_{\boldsymbol{y}}(\theta,\tau)^2} + \frac{1-2\nu}{R_{\boldsymbol{y}}(\theta,\tau) - y_2}\Big),$$

$$T_2(\theta, \boldsymbol{y}; \tau) = -\frac{R(\sin\theta - \sin\tau) - y_1 \sin\theta}{4\pi\mu R_{\boldsymbol{y}}(\theta,\tau)} \Big(\frac{y_2}{R_{\boldsymbol{y}}(\theta,\tau)^2} + \frac{1-2\nu}{R_{\boldsymbol{y}}(\theta,\tau) - y_2}\Big),$$

$$4\pi\mu T_3(\theta, \boldsymbol{y}; \tau) = -\frac{1}{R_{\boldsymbol{y}}(\theta,\tau)} \Big(\frac{y_2^2}{R_{\boldsymbol{y}}(\theta,\tau)^2} + 2(1-\nu)\Big),$$

$$R_{\boldsymbol{y}}(\theta,\tau)^2 = 4R^2(1 + R^{-1}y_1)\sin^2[(\theta-\tau)/2] + |\boldsymbol{y}|^2, \qquad |\boldsymbol{y}| = (y_1^2 + y_2^2)^{1/2}.$$

We introduce the projections of vector (2.1) onto the unit vectors $\boldsymbol{t}(\theta)$, $\boldsymbol{e}_1(\theta)$, and $\boldsymbol{e}_2$ (see Fig. 1):

$$V_t(P; \theta, \boldsymbol{y}) = \int_{-\pi}^{\pi} P(\tau)[-T_1(\theta, \boldsymbol{y}; \tau)\sin\theta + T_2(\theta, \boldsymbol{y}; \tau)\cos\theta] R\, d\tau,$$

$$V_1(P; \theta, \boldsymbol{y}) = \int_{-\pi}^{\pi} P(\tau)[-T_1(\theta, \boldsymbol{y}; \tau)\cos\theta - T_2(\theta, \boldsymbol{y}; \tau)\sin\theta] R\, d\tau, \tag{2.3}$$

$$V_2(P; \theta, \boldsymbol{y}) = \int_{-\pi}^{\pi} P(\tau) T_3(\theta, \boldsymbol{y}; \tau) R\, d\tau.$$

We assume that the function P is continuous, and its derivative is piecewise continuous and has discontinuities of the first kind (these conditions hold below). Using the methods of asymptotic analysis [6, 7, 9, 10] of integrals of the type (2.3), we obtain the representations

$$4\pi\mu V_t(P; \theta, \boldsymbol{y}) = (1-2\nu)\ \text{p.v.} \int_{\theta-\pi}^{\theta+\pi} P(\tau) \frac{\cos[(\tau-\theta)/2]}{\sin[(\tau-\theta)/2]}\, d\tau + \ldots,$$

$$4\pi\mu V_1(P; \theta, \boldsymbol{y}) = -\frac{2y_1y_2}{|\boldsymbol{y}|^2} P(\theta) + 2(1-2\nu)\,\mathrm{sign}\,(y_1)\Big(\arctan\frac{|y_2|}{|y_1|} - \frac{\pi}{2}\Big) P(\theta) + \frac{1-2\nu}{2R} Q + \ldots, \tag{2.4}$$

$$4\pi\mu V_2(P; \theta, \boldsymbol{y}) = 4(1-\nu)\Big[P(\theta)\ln\frac{|\boldsymbol{y}|}{8R} - (\boldsymbol{J}P)(\theta)\Big] - \frac{2y_2^2}{|\boldsymbol{y}|^2} P(\theta) + \ldots$$

as $|\boldsymbol{y}| \to 0$. Here p.v. is the principal value of the integral in the Cauchy sense, Q is the resultant of the loads pressing the punch against the surface of the elastic base, and $\boldsymbol{J}$ is the integral operator:

$$(\boldsymbol{J}P)(\theta) = \frac{1}{2}\int_{-\pi}^{\pi} \frac{P(\tau) - P(\theta)}{2|\sin[(\tau-\theta)/2]|}\, d\tau, \qquad Q = \int_{-\pi}^{+\pi} P(\tau) R\, d\tau. \tag{2.5}$$

Analysis of the estimates of the terms discarded in relations (2.4) is beyond the scope of this paper; we note only that they significantly depend on the differential properties of the density P [8–10].

3. Internal Asymptotic Representation. In the neighborhood of the punch base, we pass to the local coordinate system (θ, y_1, y_2), changing the variables (2.2) in relations (1.1)–(1.4) written in cylindrical coordinates. We introduce the "fast" variables

$$\boldsymbol{\eta} = (\eta_1, \eta_2) = \varepsilon^{-1}(y_1, y_2).$$

Retaining the leading (relative to the parameter ε) terms in the resulting relations, we formulate the problem for the internal asymptotic representation of the solution of the initial problem in the half-plane $\eta_2 \leqslant 0$:

$$\boldsymbol{w}(\theta, \boldsymbol{\eta}) = w_t(\theta, \boldsymbol{\eta})\boldsymbol{t}(\theta) + W_1(\theta, \boldsymbol{\eta})\boldsymbol{e}_1(\theta) + W_2(\theta, \boldsymbol{\eta})\boldsymbol{e}_2. \tag{3.1}$$

By virtue of the vector equation (1.1), we obtain the relations

$$\begin{aligned}\mu\Delta_\eta W_1(\theta,\boldsymbol{\eta})+(\lambda+\mu)[\partial_1^2 W_1(\theta,\boldsymbol{\eta})+\partial_1\partial_2 W_2(\theta,\boldsymbol{\eta})]=0,\\ \mu\Delta_\eta W_2(\theta,\boldsymbol{\eta})+(\lambda+\mu)[\partial_1\partial_2 W_1(\theta,\boldsymbol{\eta})+\partial_2^2 W_2(\theta,\boldsymbol{\eta})]=0;\end{aligned}\tag{3.2}$$

$$\mu\Delta_\eta w_t(\theta,\boldsymbol{\eta})=0,\quad \eta_2<0,\tag{3.3}$$

where Δ_η is the two-dimensional Laplace operator and $\partial_i=\partial/\partial\eta_i$. Equation (1.2) implies the boundary conditions

$$\mu\partial_2 w_t(\theta,\boldsymbol{\eta})=0,\qquad \eta_2=0;\tag{3.4}$$

$$\tau_{12}(\boldsymbol{W};\boldsymbol{\eta})\equiv\mu(\partial_1 W_2(\theta,\boldsymbol{\eta})+\partial_2 W_1(\theta,\boldsymbol{\eta}))=0,\qquad \eta_2=0.\tag{3.5}$$

Formula (1.3) is replaced by

$$\tau_{22}(\boldsymbol{W};\boldsymbol{\eta})\equiv\lambda\partial_1 W_1(\theta,\boldsymbol{\eta})+(\lambda+2\mu)\partial_2 W_2(\theta,\boldsymbol{\eta})=0,\qquad \eta_2=0,\quad R<|\eta_1|.\tag{3.6}$$

The conditions of one-sided contact (1.4) can be transformed to give

$$\begin{aligned}W_2(\theta,\boldsymbol{\eta})\leqslant-\delta_0-\beta_2R\cos\theta,\qquad \tau_{22}(\boldsymbol{W};\boldsymbol{\eta})\leqslant0,\\ [W_2(\theta,\boldsymbol{\eta})+\delta_0+\beta_2R\cos\theta]\tau_{22}(\boldsymbol{W};\boldsymbol{\eta})=0,\qquad \eta_2=0,\quad -R<\eta_1<R.\end{aligned}\tag{3.7}$$

Within the framework of the method of matched expansions, formulas (3.2)–(3.7) should be supplemented by the conditions that characterize the behavior of $\boldsymbol{w}$ at infinity. The procedure of matching of the internal (3.1) and external (2.1) asymptotic representations assumes that the leading terms of the asymptotic expansions of $\boldsymbol{v}(P;\theta,\boldsymbol{y})$ and $\boldsymbol{w}(\theta,\boldsymbol{\eta})$ coincide as $|\boldsymbol{y}|\to0$ and $|\boldsymbol{\eta}|=(\eta_1^2+\eta_2^2)^{1/2}\to\infty$, respectively. According to (2.4), we have

$$4\pi\mu w_t(\theta,\boldsymbol{\eta})=(1-2\nu)\,\text{p.v.}\int\limits_{\theta-\pi}^{\theta+\pi}P(\tau)\,\frac{\cos[(\tau-\theta)/2]}{\sin[(\tau-\theta)/2]}\,d\tau+o(1);\tag{3.8}$$

$$\begin{aligned}4\pi\mu W_1(\theta,\boldsymbol{\eta})&=-\frac{2\eta_1\eta_2}{|\boldsymbol{\eta}|^2}\,P(\theta)+2(1-2\nu)\,\text{sign}\,(\eta_1)\Big(\arctan\frac{|\eta_2|}{|\eta_1|}-\frac{\pi}{2}\Big)P(\theta)+\frac{1-2\nu}{2R}\,Q+o(1),\\ 4\pi\mu W_2(\theta,\boldsymbol{\eta})&=4(1-\nu)\Big[P(\theta)\Big(\ln\frac{|\boldsymbol{\eta}|}{R}-\ln\frac{8R}{h}\Big)-(\boldsymbol{J}P)(\theta)\Big]-\frac{2\eta_2^2}{|\boldsymbol{\eta}|^2}\,P(\theta)+o(1)\end{aligned}\tag{3.9}$$

as $|\boldsymbol{\eta}|\to\infty$.

One can see that, for a "plane" boundary layer, problem (3.2)–(3.9) splits into two problems: the two-dimensional problem of the theory of elasticity which comprises relations (3.2), (3.5)–(3.7), and (3.9) and the problem of "antiplane" shear (3.3), (3.4), and (3.8), the dependence on the "slow" angular coordinate θ being parametric.

The function w_t, which characterizes the slippage of the half-space boundary in the tangential direction, is given by

$$w_t(\theta)=\frac{(1-2\nu)(1+\nu)}{2\pi E}\,\text{p.v.}\int\limits_{\theta-\pi}^{\theta+\pi}P(\tau)\,\frac{\cos[(\tau-\theta)/2]}{\sin\,[(\tau-\theta)/2]}\,d\tau.$$

We now determine the vector $\boldsymbol{W}=(W_1,W_2)$. Since relations (3.2), (3.5)–(3.7), and (3.9) govern the two-dimensional contact problem for a punch with a rectilinear horizontal base, the following two situations are possible: either the contact condition $W_2(\theta,\eta_1,0)=-\delta_0-\beta_2R\cos\theta$ or the stress-free conditions are realized everywhere on the interval $|\eta_1|\leqslant R$.

The displacements of the points of the elastic half-plane are expressed in terms of the potential φ by the Kolosov–Muskhelishvili formula [12]

$$2\mu(W_1(\eta) + iW_2(\eta)) = æ\varphi(\eta) + \varphi(\bar\eta) - (\eta - \bar\eta)\overline{\Phi(\eta)}. \tag{3.10}$$

Here $\eta = \eta_1 + i\eta_2$, $\Phi(\eta) = \varphi'(\eta)$, and $æ = 3 - 4\nu$; the bar denotes the complex-conjugate quantity, and the prime denotes differentiation with respect to η. If the punch is loaded by a unit force, then $\hat\Phi_1(\eta) = (2\pi)^{-1}(R^2 - \eta^2)^{-1/2}$ [12] and

$$\hat\varphi_1(\eta) = -\frac{1}{2\pi i}\ln\Big(\frac{\eta}{R} + \sqrt{\frac{\eta^2}{R^2} - 1}\Big) + c. \tag{3.11}$$

Setting $c = -[4(æ + 1)]^{-1}(æ - 1)$ in (3.11) and using formula (3.10), one can construct a complex function that satisfies relations (3.1), (3.5), and (3.6) and the condition $\hat W_2(\eta) = 0$ for $|\eta_1| < R$ and $\eta_2 = 0$. Moreover, the formulas

$$4\pi\mu\hat W_1(\eta) = -\frac{2\eta_1\eta_2}{|\eta|^2} + 2(1 - 2\nu)\,\mathrm{sign}\,(\eta_1)\Big(\arctan\frac{|\eta_2|}{|\eta_1|} - \frac{\pi}{2}\Big) + O(|\eta|^{-1}),$$

$$4\pi\mu\hat W_2(\eta) = 4(1 - \nu)\ln\frac{2|\eta|}{R} - \frac{2\eta_2^2}{|\eta|^2} + O(|\eta|^{-1})$$

are valid as $|\eta| \to \infty$.

By construction, the functions

$$W_1(\theta, \boldsymbol\eta) = P(\theta)\hat W_1(\eta) + \frac{(1 - 2\nu)(1 + \nu)}{4\pi ER}Q; \tag{3.12}$$

$$W_2(\theta, \boldsymbol\eta) = P(\theta)\hat W_2(\eta) - \frac{2(1 - \nu^2)}{\pi E}\Big[P(\theta)\ln\frac{16R}{h} + (\boldsymbol J P)(\theta)\Big] \tag{3.13}$$

satisfy relations (3.2), (3.5), (3.6), and (3.9). The contact pressure determined by the principal term has the form

$$p(\theta, y_1) = \frac{P(\theta)}{\pi\sqrt{h^2 - y_1^2}}. \tag{3.14}$$

Below, we inquire whether the constraint (3.7) can be imposed on W_1 and W_2.

4. Resulting Variational Inequality and the Principal Term in the Logarithmic Asymptotic Representation of Its Solution. Let $P(\theta) > 0$. By virtue of (3.14), the second relation in (3.7) becomes an exact inequality and the first relation becomes an equality. Therefore, according to (3.13), we obtain

$$P(\theta) > 0 \;\Rightarrow\; P(\theta)\ln\frac{16R}{h} + \frac{1}{2}\int\limits_{-\pi}^{\pi}\frac{P(\tau) - P(\theta)}{2|\sin[(\tau - \theta)/2]|}\,d\tau = \frac{\pi E}{2(1 - \nu^2)}\,(\delta_0 + \beta_2 R\cos\theta). \tag{4.1}$$

If $P(\theta) = 0$, the functions (3.12) and (3.14) are independent of the "fast" variables and

$$P(\theta) = 0 \;\Rightarrow\; \frac{1}{2}\int\limits_{-\pi}^{\pi}\frac{P(\tau)}{2|\sin[(\tau - \theta)/2]|}\,d\tau \geqslant \frac{\pi E}{2(1 - \nu^2)}\,(\delta_0 + \beta_2 R\cos\theta). \tag{4.2}$$

Finally, the situation where $P(\theta) < 0$ is impossible, since it contradicts the second condition in (3.7), i.e., the following requirement arises

$$P(\theta) \geqslant 0, \qquad \theta \in (-\pi, \pi]. \tag{4.3}$$

The function P is determined from relations (4.1)–(4.3) and involves all the assumptions in the external and internal asymptotic representations of the solution of the initial problem.

For simplicity, we introduce the notation

$$\Lambda = \ln \frac{16R}{h}, \qquad \gamma(\theta) = \frac{2(1-\nu^2)}{\pi E R} P(\theta), \qquad \varphi(\theta) = \frac{\delta_0}{R} + \beta_2 \cos\theta. \tag{4.4}$$

In accordance with the general scheme for construction of the variational inequalities (see, e.g., [1]), we collect formulas (4.1)–(4.3), thus formulating the problem for the desired density γ. We multiply the equality in (4.1) and the inequality in (4.2) by an arbitrary smooth nonnegative function σ and integrate over Γ. With allowance for (4.4), we obtain

$$\Lambda(\gamma,\sigma) + (\boldsymbol{J}\gamma,\sigma) \geqslant (\varphi,\sigma), \qquad (\gamma,\sigma) = \int_{-\pi}^{\pi} \gamma(\tau)\sigma(\tau)\,d\tau. \tag{4.5}$$

Setting $\sigma = \gamma$, we repeat the procedure. By virtue of (4.1) and (4.2), we find

$$\Lambda(\gamma,\gamma) + (\boldsymbol{J}\gamma,\gamma) = (\varphi,\gamma). \tag{4.6}$$

Subtracting (4.5) from (4.6), we obtain the formula

$$\Lambda(\gamma,\gamma-\sigma) + (\boldsymbol{J}\gamma,\gamma-\sigma) \leqslant (\varphi,\gamma-\sigma) \qquad \forall\,\sigma \geqslant 0. \tag{4.7}$$

The resulting variational inequality is formulated as the problem of determining the nonnegative function γ that satisfies relation (4.7) for an arbitrary smooth nonnegative function σ. We note that the solution of problem (4.7) which possesses a certain smoothness (if it exists) satisfies (4.1)–(4.3); these formulas are derived from (4.7) according to the general scheme [1].

Remark 2. By virtue of the definite properties of the operator $\boldsymbol{J}$ [8], the question of solvability of problem (4.7) remains open. However, the asymptotic solution (4.7) is sufficient for construction of the asymptotic representation of the solution of problem (1.1)–(1.5).

Problem (4.7) contains a large parameter. Ignoring the second term on the left side of (4.7), we obtain

$$\Lambda(\gamma_1,\gamma_1-\sigma) \leqslant (\varphi,\gamma_1-\sigma) \qquad \forall\sigma \geqslant 0. \tag{4.8}$$

Let γ_1 be the solution of the variational inequality (4.8). We substitute the function $\sigma = \gamma_1 + \sigma_1$ with arbitrary $\sigma_1 \geqslant 0$ as a trial function into (4.8) and obtain the inequality $(\Lambda\gamma_1 - \varphi, \sigma_1) \geqslant 0\ \forall\sigma_1 \geqslant 0$. Hence,

$$\Lambda\gamma_1(\theta) - \varphi(\theta) \geqslant 0, \qquad \theta \in (-\pi,\pi]. \tag{4.9}$$

We first set $\sigma = 0$ and then $\sigma = 2\gamma_1$ in (4.8). As a result, we have

$$(\Lambda\gamma_1(\theta) - \varphi(\theta))\gamma_1(\theta) = 0, \qquad \theta \in (-\pi,\pi]. \tag{4.10}$$

Using the notation (4.4), we rewrite (4.9) and (4.10) in the alternative form and add the condition of nonnegative desired density:

$$P_1(\theta) > 0 \Rightarrow \quad P_1(\theta)\ln\frac{16R}{h} = \frac{\pi E}{2(1-\nu^2)}(\delta_0 + \beta_2 R\cos\theta); \tag{4.11}$$

$$P_1(\theta) = 0 \Rightarrow \quad \delta_0 + \beta_2 R\cos\theta \leqslant 0; \tag{4.12}$$

$$P_1(\theta) \geqslant 0, \qquad \theta \in (-\pi,\pi]. \tag{4.13}$$

Let $(t)_+ = (t + |t|)/2$. As a result, the function P_1, which satisfies (4.11)–(4.13) and, hence, is the solution of problem (4.8), has the form

$$P_1(\theta) = \frac{\pi E}{2(1-\nu^2)}\,\frac{1}{\ln(16R/h)}\,(\delta_0 + \beta_2 R\cos\theta)_+. \tag{4.14}$$

Relations (4.11)–(4.13) formalize the problem of one-sided contact for a Winkler elastic base.

5. Contact-Zone Variation. For the initial problem, we assume that $\beta_2 > 0$ and $-1 \leqslant -\delta_0/(\beta_2 R) < 1$. In this case, we have $P_1(\theta) > 0$, provided the magnitude of the angle θ is bounded by the value

$$\Theta_1 = \arccos\,[-\delta_0/(\beta_2 R)]. \tag{5.1}$$

The contact problems can be solved by an iterative method which consists of solving the problem with a fixed contact zone in each iteration. In the transition from one iteration to another, the points at which the contact pressure is negative are excluded from the region of contact. At the beginning of the process, it is necessary that the assumed region of contact enclose a real one. Argatov and Nazarov [8] showed that the desired narrow zone of contact is a small perturbation of the arc $\theta \in (-\Theta_1, \Theta_1)$. One should expect that formula (4.14) gives the upper bound (see Remark 1) and the improvement of the first-order approximation decreases the quantity (5.1).

We set $\gamma_n(\theta) \equiv 0$ for $\theta \notin (-\Theta_{n-1}, \Theta_{n-1})$. In accordance with (2.5), we have

$$(\boldsymbol{J}\gamma_n)(\theta) = \frac{1}{2} \int\limits_{-\Theta_{n-1}}^{\Theta_{n-1}} \frac{\gamma_n(\tau) - \gamma_n(\theta)}{2|\sin[(\tau-\theta)/2]|}\, d\tau$$

$$- \frac{\gamma_n(\theta)}{2} \Big(\int\limits_{-\pi}^{-\Theta_{n-1}} \frac{1}{2|\sin\,[(\tau-\theta)/2]|}\, d\tau + \int\limits_{\Theta_{n-1}}^{\pi} \frac{1}{2|\sin\,[(\tau-\theta)/2]|}\, d\tau \Big).$$

Calculating these integrals and substituting the result into (4.8), we obtain

$$\gamma_n(\theta) \Big(\ln\frac{16}{\varepsilon} + \frac{1}{2} \ln\Big[\tan\frac{\Theta_{n-1}+\theta}{4}\, \tan\frac{\Theta_{n-1}-\theta}{4}\Big]\Big) + \frac{1}{2} \int\limits_{-\Theta_{n-1}}^{\Theta_{n-1}} \frac{\gamma_n(\tau) - \gamma_n(\theta)}{2|\sin\,[(\tau-\theta)/2]|}\, d\tau = \varphi(\theta) \tag{5.2}$$

for $\theta \in (-\Theta_{n-1}, \Theta_{n-1})$.

We write the solution of Eq. (5.2) in the form of the series

$$\gamma_n(\theta) \sim \Lambda^{-1}\gamma_n^1(\theta) + \Lambda^{-2}\gamma_n^2(\theta) + \ldots, \qquad \gamma_n^1(\theta) = \varphi(\theta),$$
$$\gamma_n^2(\theta) = -\frac{1}{2}\,\varphi(\theta) \ln\Big[\tan\frac{\Theta_{n-1}+\theta}{4}\, \tan\frac{\Theta_{n-1}-\theta}{4}\Big] - \frac{1}{2} \int\limits_{-\Theta_{n-1}}^{\Theta_{n-1}} \frac{\varphi(\tau) - \varphi(\theta)}{2|\sin\,[(\tau-\theta)/2]|}\, d\tau. \tag{5.3}$$

Retaining two terms in the expansion (5.3), we obtain the second approximation to the solution of problem (4.1)–(4.3):

$$P_2(\theta) = \frac{\pi E}{2(1-\nu^2)}\, \frac{1}{\Lambda} \Big(\delta_0 + \beta_2 R\cos\theta + \frac{\beta_2 R}{\Lambda}\Big[\cos\theta - \cos\frac{\Theta_1}{2}\, \cos\frac{\theta}{2}\Big]$$
$$- \frac{\delta_0 + \beta_2 R\cos\theta}{2\Lambda} \ln\Big[\tan\frac{\Theta_1+\theta}{4}\, \tan\frac{\Theta_1-\theta}{4}\Big]\Big)_+, \tag{5.4}$$

which improves (4.11).

The contact arc $\theta \in (-\Theta_2, \Theta_2)$ corresponds to the line-pressure distribution density (5.4). It is noteworthy that the equality $\Theta_2 = \Theta_1 - \Lambda^{-1} S_1$ ensures the same accuracy as that in (5.4). Ignoring the quantities $O(\Lambda^{-2}\ln\Lambda)$, we obtain the following transcendental equation for the contact-zone variation:

$$S_1\Big[1 - \frac{1}{2\Lambda} \ln\Big(\frac{S_1}{4\Lambda}\tan\frac{\Theta_1}{2}\Big)\Big] = \frac{1}{2}\tan\frac{\Theta_1}{2}.$$

Its solution has the asymptotic representation $S_1 = (1/2)\tan\,(\Theta_1/2) + O(\Lambda^{-1}\ln\Lambda)$ (the subtrahend in square brackets is ignored). This formula fails for values of Θ_1 close to π, i.e., it does not describe detachment of the punch from the base.

If $\Theta_1^0 = \pi$, which corresponds to $\delta_0 = \beta_2 R$, Eq. (5.2) coincides with (4.1) and admits a closed-form solution [10]. Retaining, as before, two terms in its logarithmic asymptotic representation, we obtain

$$P_2^0(\theta) = \frac{\pi E}{2(1-\nu^2)} \frac{\beta_2 R}{\Lambda}\Big(1 + \cos\theta + \frac{2}{\Lambda}\cos\theta\Big)_+.$$

The angle of the contact arc is determined from the equation $\Lambda(1 + \cos\Theta_2^0) + 2\cos\Theta_2^0 = 0$. Its solution is written in the finite form and has the asymptotic representation $\Theta_2^0 = \pi - 2\Lambda^{-1/2} + O(\Lambda^{-3/2})$.

Conclusions. The formulas obtained should be understood precisely in the asymptotic sense: they are the more exact the smaller the parameters ε and Λ^{-1} [$\Lambda = \ln(16/\varepsilon)$]. In the neighborhood of the contact-zone ends, the stress-strain state of the elastic body is three-dimensional and, hence, it cannot be described by the "plane" formula (3.14). To determine the contact pressures in these regions, one can combine numerical and asymptotic methods. The constructed formulas do not allow one to determine the asymptotic representation of the contact zone with an accuracy comparable with the half-width of the punch $h = \varepsilon R$. Only owing to the new scaling factor $\Lambda^{-1}R$ that appears in the resulting problem does it become possible to study the question of the contact-arc variation.

We mention some ways of generalization of the above-considered one-dimensional model of one-sided contact. Replacement of the elastic base by a layer of finite thickness complicates the model insignificantly. For a punch of variable thickness with a "wavy" base and a noncircular middle line, the problem can be studied with the use of the results of [6, 8, 10].

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REFERENCES

1. R. Glowinski, J. L. Lions, and R. Tremolieres, *Analyse Numerique des Inequations Variationnelles*, Dunond, Paris (1976).
2. A. S. Kravchuk, *Variational and Quasivariational Inequalities in Mechanics* [in Russian], Izd. MGAPI, Moscow (1997).
3. R. V. Gol'dshtein and A. A. Spektor, "Variational estimates of the solutions of some mixed spatial problems of the theory of elasticity with an unknown boundary," *Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela*, No. 2, 82–94 (1978).
4. V. A. Kovtunenko, "Numerical method of solving the problem of contact of an elastic plate with an obstacle," *Prikl. Mekh. Tekh. Fiz.*, **35**, No. 5, 142–146 (1994).
5. A. M. Khludnev, "Contact of two plates one of which contains a crack," *Prikl. Mat. Mekh.*, **61**, No. 2, 882–894 (1997).
6. S. A. Nazarov and O. R. Polyakova, "Failure of a narrow bridge between coplanar cracks," *Prikl. Mat. Mekh.*, **55**, No. 1, 157–165 (1991).
7. Kalker J. J., "On elastic line contact," *Trans. ASME, Ser. E, J. Appl. Mech.*, **39**, No. 4. P. 1125–1132 (1972).
8. I. I. Argatov and S. A. Nazarov, "Asymptotic solution of the Signorini problem with an obstacle on a thin oblong set," *Mat. Sb.*, **187**, No. 10, 3–32 (1996).
9. M. V. Fedoryuk, "Asymptotic behavior of the solution of the Dirichlet problem for the Laplace and Helmholtz equations in the exterior of a thin cylinder," *Izv. Akad. Nauk SSSR, Ser. Mat.*, **46**, No. 1, 167–186 (1981).
10. V. G. Maz'ya, S. A. Nazarov, and B. A. PLaménevskii, "Asymptotic behavior of the solutions of the Dirichlet problem for a three-dimensional region with a cut thin tube," *Mat. Sb.*, **116**, No. 2, 187–217 (1981).
11. A. M. Il'in, *Matching the Asymptotic Expansions of the Solutions of Boundary-Value Problems* [in Russian], Nauka, Moscow (1989).
12. N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, Noordhoff, Groningen, Holland (1953).